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1

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A "DYNAMIC" PROOF OF THE FROBENIUS-PERRON THEOREM FOR METZLER MATRICES

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Matrices with non-negative off-diagonal elements have many applications in mathematical economics and other fields of investigation. Economists have called them "Metzler matrices", because of their study by L. Metzler (1945). An important property, especially for the study of stability of dynamic systems, is that the largest real part of the characteristic roots is itself a characteristic root and has a semi-positive characteristic vector.

There is a less well-known property of linear dynamic systems governed by Metzler matrices: if the forcing term is a non-negative vector and if the system starts in the positive orthant, it will remain there forever. The proposition seems to have been first proved by Samuel Karlin, ~~though~~ never published by him; his proof is referred to by Beckenbach and Bellman (1961), p. 137.

Karlin's result does not appear to be derivable from the standard Frobenius-Perron theorem (~~Theorem~~ 4 below). Its proof is not very hard, however. The question is then raised, whether the Frobenius-Perron result is derivable simply from ~~Karlin's~~ theorem. This note shows that the answer is affirmative. The result may very possibly be useful for expository purposes.

I start by demonstrating Karlin's theorem, for completeness. I then derive several familiar properties of Metzler matrices from which the Frobenius-Perron theorem can be derived.

DEF. 1: *A is a* Metzler matrix *if $a_{ij} \geq 0$ for all $i \neq j$.*

There are several ways of proving Karlin's theorem. In the following, I use the concept of the exponential of a matrix (for a more elementary but lengthier proof, see Arrow (1960)).

DEF. 2:

$$\exp A = \Sigma_{n=0}^{\infty} A^n / n!.$$

The infinite series converges absolutely for any matrix A.

If A and B are matrices that commute (i.e., $AB = BA$), then it is easy to see that the binomial theorem is valid for $(A + B)^n$. If we substitute in the infinite series defining exp $(A + B)$ and rearrange terms, it is easy to see that

(1) if A and B commute, then $\exp(A + B) = (\exp A)(\exp B)$.

If A is non-negative, then of course $A^n \geq 0$ for all n. From Def. 2,

(2) $\exp A \geq I$ if $A \geq 0$.

Now suppose A is a Metzler matrix. By Def. 1, we can find a scalar, s such that $A - sI \geq 0$. Clearly, sI commutes with any matrix and in particular with $A - sI$. Then,

$$\exp A = \exp [sI + (A - sI)] = \exp (sI) \exp (A - sI).$$

But, from Def. 2, $\exp (sI) = e^s I$, since $I^n = I$ for all n.

(3) $\exp A = p \exp (A - sI)$ for some scalar $p > 0$.

From (2), with A replaced by $A - sI$, and (3),

(4) if A is Metzler, $\exp A \geq pI$ for some $p > 0$.

Therefore,

If A is Metzler, $(\exp A)x \geq px \gg 0$ if $x \gg 0$, and $(\exp A)x \geq px \geq 0$ if $x \geq 0$.

(I use the notations $x \geq 0$, $x > 0$, $x \gg 0$, to mean, respectively, $x_i \geq 0$ for all i, $x \geq 0$ and $x_i > 0$ for some i, and $x_i > 0$ for [illegible]

LEMMA 1. *If A is Metzler,* $(\exp A)x \gg 0$ *if* $x \gg$ [illegible] $\exp A)x \geq 0$ *if* $x \geq 0$.

Let $x(t)$ be the solution of the differential equation,

$$\dot{x} = Ax + b,$$

with given initial condition $x(0)$. Then $x(t)$ can be written,

$$x(t) = (\exp\ At)x(0) + \int_0^t [\exp\ (Au)b]du.$$

Suppose A Metzler, $b \geq 0$, and $x(0) \gg 0$. From Def. 1, At is Metzler for $t \geq 0$. Then, from Lemma 1,

THEOREM 1 (KARLIN). *If A is Metzler and $b \geq 0$ then any solution of the differential equation,*

$$\dot{x} = Ax + b,$$

for which $x(0) \gg 0$ has the property that $x(t) \gg 0$ for all $t \geq 0$.

Now suppose that A is a *stable* Metzler matrix, so that the real parts of all characteristic roots are negative. Take any vector y for which $Ay \leq 0$. Take any solution to the differential equation,

$$\dot{x} = A(x - y),$$

for which $x(0) \gg 0$. Since A is stable, $\lim_{t\to\infty} x(t) = y$. Since $-Ay \geq 0$ and A is Metzler, Theorem 1 implies that $x(t) \gg 0$ for all $t \geq 0$. Therefore, $y \geq 0$.

THEOREM 2. *If A is a stable Metzler matrix and $Ay \leq 0$, then $y \geq 0$.*

Suppose $Ay < 0$. Then clearly $y = 0$ is impossible.

COROLLARY 1. *If A is a stable Metzler matrix and $Ay < 0$, then $y > 0$.*

Now a sort of converse of Theorem 2 can be shown. Suppose A is a Metzler matrix and $Ay \ll 0$ for some $y > 0$. Clearly, by perturbing y, it can be assumed that,

(5) $Ay \ll 0, y \gg 0.$

Let the superscript T denote "transpose." Let $x(t)$ satisfy,

(6) $\dot{x} = A^T x, x(0) \gg 0.$

Then, by Theorem 1, $x(t) \gg 0$ for $t \geq 0$. Define,

(7) $\xi(t) = y^T x(t).$

Since $y \gg 0$ and $x(t) \gg 0$,

(8) $\xi(t) > 0$ for $t \geq 0$.

From (5),

(9) $Ay \leq my$ for some $m < 0$.

From (7) and (6),

$$\dot{\xi} = y^T \dot{x} = y^T A^T x = (Ay)^T x \leq my^T x = m\xi,$$

By integration and (8),

$$0 < \xi(t) \leq \xi(0)e^{mt} \text{ for all } t \geq 0.$$

Hence, $\xi(t)$ must approach 0 as $t \to \infty$. Since $y \gg 0$ and $x(t) \gg 0$ for all $t \geq 0$, this is possible only if $x(t)$ approaches 0.

Thus, every solution of the differential equation,

$$\dot{x} = A^T x,$$

for which $x(0) \gg 0$, approaches 0. But then every solution approaches 0 so that A^T and therefore A is stable.

THEOREM 3. *If A is Metzler and $Ay \ll 0$ for some $y > 0$, then A is stable.*

We can now deduce the existence of a dominant root and semi-positive dominant vector from Corollary 1 and Theorem 3.

DEF. 3: *$\sigma(A)$ = maximum of real parts of characteristic roots of A.*

For any $s > \sigma(A)$, $A - sI$ is stable. Let A, in particular, be Metzler. Then $A - sI$ is Metzler and stable. It is certainly non-singular. Choose a fixed vector

(10) $c \ll 0$.

For each $s > \sigma(A)$, we can define $y(s)$ as satisfying

(11) $(A - sI)y(s) = c$.

Then by Corollary 1,

(12) $y(s) > 0$ for all $s > \sigma(A)$.

Let e be the vector all of whose components are 1. Then, $e^T y(s) > 0$. Define

(13) $\eta(s) = [e^T y(s)]^{-1}, x(s) = \eta(s)\ y(s)$,

so that

(14) $x(s)$ belongs to the unit simplex, S, for all $s > \sigma(A)$.

From (11) and (13),

(15) $(A - sI)x(s) = \eta(s)c$ for $s > \sigma(A)$.

From (14), the function, $x(s)$ has a limit point, say x, as s approaches $\sigma(A) + 0$. Hence, $x \epsilon S$, and, in particular,

(16) $x > 0$.

By definition, there exists a sequence $\{s_n\}$ such that,

$$\lim_{n\to\infty} s_n = \sigma(A) + 0,$$
$$\lim_{n\to\infty} x(s_n) = x.$$

If we substitute s_n for s in (15) and let n approach ∞, then the left-hand side converges, and therefore the right-hand side must also converge, so that,

$$\lim_{n\to\infty} \eta(s_n) = \eta \geq 0.$$

From (15), with s replaced by s_n and n approaching infinity,

$$(17) \quad [A - \sigma(A)I]x = \eta c.$$

Suppose $\eta > 0$. Then $[A - \sigma(A)I]x \ll 0$. From (16) and Theorem 3,

$$0 > \sigma[A - \sigma(A)I] = \sigma(A) - \sigma(A) = 0,$$

a contradiction. Hence $\eta = 0$, so that, from (17),

$$Ax = \sigma(A)x.$$

THEOREM 4. *If A is a Metzler matrix, there exists a real number, σ and a real vector $x > 0$ such that, (a) $Ax = \sigma x$, and (b) for every characteristic root, λ, of $A, R(\lambda) \leq \sigma$.*

The usual criteria for stability and for existence of non-negative solutions to the equation $Ax = b$ for Metzler matrices are all either all in the theorems already stated or can easily be deduced from them.

APPENDIX

As is well known, Theorem 4 can be strengthened in one way: conclusion (b) can be replaced by,

(b') for every characteristic root, λ, of A, with $\lambda \neq \sigma, R(\lambda) < \sigma$.

In view of (b), this is equivalent to the statement,

(b'') if $\sigma + i\tau$ (τ real) is a characteristic root of A, then $\tau = 0$.

I now prove statement (b'') using Theorems 1 and 4. Without loss of generality, it can be assumed that

(18) $\sigma(A) = 0$.

Suppose (b'') false for some A. Then, under (18),

(19) $i\tau$ is a characteristic root of A for some $\tau \neq 0$.

If A is a matrix of order n, let S be any subset of the integers $1, \ldots, n$, and S^* its complement.

DEF. 4. *A is* decomposable *if there exists a non-empty set S with non-empty complement such that $A_{SS^*} = 0$. Otherwise A is* indecomposable.

It is easy to see that if (18) and (19) hold for some Metzler matrix, they hold for some indecomposable Metzler matrix. By successive decompositions, we can find a partition of the integers $1, \ldots, n$, into sets $S(i)(1 = 1, \ldots, p)$ such that,

$$A_{S(i)S(j)} = 0 \text{ for } i < j.$$

$$A_{S(i)S(i)} \text{ is indecomposable for each } i.$$

Then any characteristic root of A is a characteristic root of $A_{S(i)S(i)}$ for some i, and conversely. From (19), $i\tau$ is a characteristic root of $A_{S(i)S(i)}$ for some i. By Def. 3, $\sigma(A_{S(i)S(i)}) \geq 0$. But, by Theorem 4, $\sigma(A_{S(i)S(i)})$ is a root of $A_{S(i)S(i)}$ and therefore is a root of A, so that,

$$\sigma(A_{S(i)S(i)}) \leq \sigma(A) = 0,$$

and therefore $\sigma(A_{S(i)S(i)}) = 0$. Hence, (18) and (19) hold for $A_{S(i)S(i)}$ which is indecomposable and Metzler, by definition, and we can say that they hold for some matrix A which also satisfies,

(20) A indecomposable.

DEF. 5. *For any vector* $x, Z(x) = \{i | x_i = 0\}$.

We abbreviate $Z(x)$ as Z, when the context makes it clear.

The interesting implication of (20) for the present purpose is,

LEMMA 2. *If A is indecomposable and Metzler, $x > 0$, and $Z(x)$ is non-empty, the $(Ax)_Z \neq 0$.*

PROOF: Since $x > 0$, Z^* is non-empty. By Def. 5, $x_Z = 0, x_{Z^*} \gg 0$. Hence,

$$(Ax)_Z = A_{ZZ}x_Z + A_{ZZ^*}x_{Z^*} = A_{ZZ^*}x_{Z^*},$$

Since A is Metzler, all elements of A_{ZZ^*} are non-negative. Since A is indecomposable, at least one element of A_{ZZ^*} is non-zero and therefore positive. Since $x_{Z^*} \gg 0$, it must be that $(Ax)_Z > 0$.

From (18) and Theorem 4, there exists $x^* > 0$ such that $Ax^* = 0$. If $Z(x^*)$ were non-empty, then certainly $(Ax^*)_Z = 0$, in contradiction to Lemma 2. Hence, $Z(x^*)$ is empty.

(21) There exists $x^* \gg 0$ such that $Ax^* = 0$.

Let $u + iv$ (u, v real) be a characteristic vector of A corresponding to the root ir. Then it is easy to see that u is any vector satisfying.

(22) $A^2 u = -r^2 u, u \neq 0$.

Then the solution of the differential equation.

(23) $\dot{x} = Ax$,

with initial condition $x(0) = u$ is,

(24) $x'(t) = u \cos \tau t + (1/\tau) Au \sin \tau t.$

If $u \geq 0$, then, by Theorem 1, $x'(t) \geq 0$ for all $t \geq 0$, and in particular for $t = \pi/\tau$. But $x'(\pi/\tau) = -u$, so that $u = 0$, a contradiction. Hence, it is impossible that $u \geq 0$. Since $-u$ also satisfies (22), it cannot be that $-u \geq 0$.

(25) If u satisfies (22), then $u_i > 0$ for some i, $u_j < 0$ some j.

For any x^* satisfying (21), the solution to (23) with $x(0) = x^* + u$ is,

(26) $x(t) = x^* + x'(t).$

From (21) and (25), x^* can be chosen so that $x^* + u \geq 0$, with $(x^* + u)_i = 0$ for at least one i, so that $Z(x^* + u)$ is non-empty. Since $u_i > 0$ for some i, by (25), $(x^* + u)_i > 0$ and therefore $x^* + u > 0$. Then $x(t) \geq 0$ for all $t \geq 0$; but since $x'(t)$ is periodic, in fact, $x(t) \geq 0$ for all t. If $i \epsilon Z(x^* + u)$, then $x_i(0) = 0, x_i(t) \geq 0$ for all t; therefore, $\dot{x}_i(0) = 0$ for all $i \epsilon Z$. From the differential equation (23), $[Az(0)]_Z = 0$, in contradiction to Lemma 2. Hence, the existence of a purely imaginary characteristic root leads to a contradiction.

THEOREM 4'. *If A is Metzler, then there exists a real number σ and a real vector $x > 0$ such that, (a) $Ax = \sigma x$, and (b) $R(\lambda) < \sigma$ if λ is any characteristic root of A with $\lambda \neq \sigma$.*

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